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TWICE-PUNCTURED HYPERBOLIC SPHERE WITH A CONICAL SINGULARITY AND GENERALIZED ELLIPTIC INTEGRAL

G. D. ANDERSON, T. SUGAWA, M. K. VAMANAMURTHY, AND M. VUORINEN

ABSTRACT. We describe, in terms of generalized elliptic integrals, the hyperbolic metric of the twice-punctured sphere with one conical singularity of prescribed order. We also give several monotonicity properties of the metric and a couple of applications.

1. Introduction

The hyperbolic metric $\rho(z)|dz|$ on the thrice-punctured sphere $\widehat{\mathbb{C}}\setminus\{0,1,\infty\}$ is one of the fundamental tools in complex analysis. Indeed, for instance, the big Picard theorem can be derived by a careful look at the metric $\rho(z)|dz|$ and the distance induced by it. It is known that the density function $\rho(z)$ can be expressed explicitly as

$$\rho(z) = \frac{\pi}{8|z(1-z)|\operatorname{Re}\{\mathrm{K}(z)\mathrm{K}(1-\bar{z})\}},$$

where $\mathrm{K}(z)$ is the complete elliptic integral of the first kind given in (3.1) (see [2] or [15]). On the other hand, it has been recognized that generalized elliptic integrals $\mathrm{K}_a(z)$ and $\mathrm{E}_a(z)$, defined in (3.2) and (3.3) respectively, share many properties with the original complete elliptic integrals (cf. [4]).

In the present paper, it is shown that the hyperbolic metric of a twice-punctured sphere with one conical singularity of prescribed angle can be expressed in terms of these generalized complete elliptic integrals.

2. Hyperbolic metric with conical singularities

A hyperbolic metric of a compact Riemann surface R with conical singularities of angle $2\pi\theta_j,\ \theta_j \in [0,+\infty)\setminus\{1\}$, at points $p_j \in R,\ j = 1,\dots,N$, is a conformal metric on $R\setminus\{p_1,\dots,p_N\}$ of the form $ds = e^{\varphi(z)}|dz|$, where φ is a smooth function satisfying the Liouville equation

$$\Delta\varphi = 4e^{2\varphi} \tag{2.1}$$

on $R\setminus\{p_1,\dots,p_N\}$ and possessing the asymptotic behavior

$$\varphi(z) = \begin{cases} -(1-\theta_j)\log|z-z_j| + O(1) & \text{if } \theta_j > 0, \\ -\log|z-z_j| - \log(-\log|z-z_j|) + O(1) & \text{if } \theta_j = 0 \end{cases} \tag{2.2}$$

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as $z \to z_j = z(p_j)$, where z is a local coordinate of R around p_j. Note that a conical singularity of angle 0 is called a *puncture* or a *cusp*.

The remainder term $O(1)$ in the above is known to be continuous at $z = z_j$ by a detailed study of the local behavior of solutions to the Liouville equation at the isolated singularities by Nitsche [12] (see also [9]).

Heins [8, Chap. II] proved that for a compact Riemann surface R of genus g and finite points in it with given angles as above, a hyperbolic metric on R with the behavior described in (2.2) exists *uniquely* as long as the condition

$$2(1-g) - \sum_{j=1}^{N}(1-\theta_j) < 0 \tag{2.3}$$

is satisfied. This constraint comes from the Gauss-Bonnet formula. This result was previously known by Picard [13] when $g = 0$. Practically, this unique metric as above is called the (complete) hyperbolic metric of the Riemann surface $R \setminus \{p_1, \dots, p_n\}$ with conical singularities of angle $2\pi\theta_j$ at p_j ($j = n+1, \dots, N$), where $\theta_1 = \cdots = \theta_n = 0 < \theta_j \neq 1$ ($j = n+1, \dots, N$).

The hyperbolic metric treated in the present paper corresponds to the case when $R = \widehat{\mathbb{C}}$, $g = 0$, $N = 3$, $(p_1, p_2, p_3) = (0, 1, \infty)$ and $(\theta_1, \theta_2, \theta_3) = (0, 0, \alpha)$, where $0 \le \alpha < 1$. Note here that this case always satisfies the condition (2.3). We denote this metric by $\rho_\alpha(z)|dz|$.

When $\alpha = 0$, the metric ρ_0 is simply the usual hyperbolic metric ρ of $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$ (without conical singularities). By uniqueness of the hyperbolic metric with conical singularities, the metric admits the obvious symmetry $\rho_\alpha(z) = \rho_\alpha(1-z) = \rho_\alpha(\bar{z})$.

We remark that for a Möbius transformation M, $\rho_\alpha(M(z))|M'(z)|$ gives the density of the hyperbolic metric of $\widehat{\mathbb{C}} \setminus \{M(0), M(1)\}$ with a conical singularity of angle $2\pi\alpha$ at $M(\infty)$. For instance, the hyperbolic metric $\tilde{\rho}_\alpha(z)|dz|$ of the twice-punctured sphere $\widehat{\mathbb{C}} \setminus \{1, \infty\} = \mathbb{C} \setminus \{1\}$ with a conical singularity of angle $2\pi\alpha$ at 0 can be obtained by $\tilde{\rho}_\alpha(z) = \rho_\alpha(1/z)/|z|^2$.

3. Generalized elliptic integrals

The complete elliptic integrals of the first and the second kind are defined, respectively, by

$$\mathrm{K}(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}} \quad \text{and} \quad \mathrm{E}(z) = \int_0^1 \sqrt{\frac{1-zt^2}{1-t^2}}\,dt. \tag{3.1}$$

Note that these functions can be expressed also by the hypergeometric function:

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \quad |z| < 1,$$

where $(a,0)=1$ and $(a,n)=a(a+1)(a+2)\cdots(a+n-1)$ for $n\ge 1$, and $c\neq 0,-1,-2,\dots$. Indeed,

$$\mathrm{K}(z)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;z\right)\quad\text{and}\quad \mathrm{E}(z)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;z\right).$$

Let $0<a<1$. The generalized complete elliptic integrals of the first and the second kind with signature $1/a$ are defined, respectively, by

$$\begin{aligned}\mathrm{K}_a(z)&=\frac{\pi}{2}F(a,1-a;1;z)\\&=\sin(\pi a)\int_0^1\frac{t^{1-2a}dt}{(1-t^2)^{1-a}(1-z\,t^2)^a}\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}\mathrm{E}_a(z)&=\frac{\pi}{2}F(a-1,1-a;1;z)\\&=\sin(\pi a)\int_0^1\left(\frac{1-z\,t^2}{1-t^2}\right)^{1-a}t^{1-2a}dt.\end{aligned}\tag{3.3}$$

Here, note that $\mathrm{K}_a(z)$ and $\mathrm{E}_a(z)$ are defined as (single-valued) analytic functions in $z\in\mathbb{C}\setminus[1,+\infty)$.

We remark that the above definition is slightly different from the usual one. The (traditional) generalized complete elliptic integrals of the first and the second kind usually refer to $\mathrm{K}_a(x^2)$ and $\mathrm{E}_a(x^2)$ for $0<x<1$ in our notation.

We mean by K_a' and E_a' the derivatives of K_a and E_a, though these are often used to mean the complementary functions. For the complementary functions, we adopt the notation $\mathrm{K}_a^*(z)=\mathrm{K}_a(1-z)$ and $\mathrm{E}_a^*(z)=\mathrm{E}_a(1-z)$ in the present paper.

The following formula, which is a special case of Elliott's identity (see [4]), will be used at a crucial step in the computation of the hyperbolic metric:

$$\mathrm{K}_a^*(z)\mathrm{E}_a(z)+\mathrm{E}_a^*(z)\mathrm{K}_a(z)-\mathrm{K}_a^*(z)\mathrm{K}_a(z)=\frac{\pi\sin(\pi a)}{4(1-a)}.\tag{3.4}$$

Some information about the behavior of the hypergeometric function near $z=x=1$ will be needed below. The following result can be found in Chapter 15 of the book [1].

$$\begin{cases}F(a,b;c;1-)=\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c>a+b,\\ F(a,b;a+b;x)=\dfrac{1}{B(a,b)}\log\dfrac{1}{1-x}+O(1),\\ \qquad\qquad(\text{as } x\to 1-),\\ F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x),\ c<a+b.\end{cases}\tag{3.5}$$

Here $B(a,b)$ denotes the beta function.

4. Computation of $\rho_\alpha(z)$

A relation between conformal mappings and the generalized complete elliptic integral $\mathrm{K}_a(z)$ of the first kind is given by [4, Theorem 2.2] when the argument z is real and between 0 and 1. We will now give another aspect of $\mathrm{K}_a(z)$ for the complex argument z.

As is stated in [16, Lemma 2], the hyperbolic metric of the sphere with given conical singularities can be described in terms of solutions to a second-order Fuchsian differential equation with regular singularities at the cone points. In our case, the metric is described explicitly in terms of generalized elliptic integrals.

We begin with a general case. Let a, b, and c be real numbers. It is a classical fact that the function $f(z) = i\,F(a,b;a+b+1-c;1-z)/F(a,b;c;z)$ maps the upper half plane $\mathbb{H}$ onto a curvilinear triangle bounded by three circular arcs and having the interior angles $(1-c)\pi$ at $f(0)$, $(c-a-b)\pi$ at $f(1)$, and $(b-a)\pi$ at $f(\infty)$, provided that these angles are all nonnegative and the sum is less than π (see, for instance, [11, pp. 206, 207]). Note that the segment $(0,1)$ of the real axis is mapped by f to a part of the imaginary axis and that f maps $\mathbb{C}\setminus((-\infty,0]\cup[1,+\infty))$ conformally onto the domain which is the union of $f((0,1))$, $f(\mathbb{H})$, and its reflection in the imaginary axis.

In the particular case when $0<a<1$, $b=1-a$, and $c=1$, the function f can be written in the form $i\,\mathrm{K}_a(1-z)/\mathrm{K}_a(z)$, and the image $f(\mathbb{H})$ is a circular triangle with interior angles 0, 0, and $|1-2a|\pi$. More specifically, we have the following result.

4.1. **Lemma.** *Let* $f_a(z) = i\,\mathrm{K}_a(1-z)/\mathrm{K}_a(z)$, $0<a<1$. *Then the image* $f_a(\mathbb{H})$ *of the upper half plane* $\mathbb{H}$ *under* f_a *is the hyperbolic triangle* Δ_a *in* $\mathbb{H}$ *whose interior angles are* 0, 0, *and* $|1-2a|\pi$ *at the vertices* 0, ∞, *and* $e^{|1-2a|\pi i/2}$, *respectively. More precisely,* $\Delta_a = \{w\in\mathbb{H} : \ 0<\operatorname{Re} w < \sin(\pi a),\ |2w\sin(\pi a)-1|>1\}$.

Proof. Since $f_a(\mathbb{H})$ is a Jordan domain, f_a extends to a homeomorphism from the closure of $\mathbb{H}$ onto the closure of $f_a(\mathbb{H})$. First note that f_a maps the interval $(0,1)$ onto the whole positive imaginary axis. Since the interior angles of $f_a(\mathbb{H})$ at $f_a(0)$ and $f_a(1)$ are both 0, the boundary arcs $f_a((1,+\infty))$ and $f_a((-\infty,0))$ are contained in hyperbolic geodesics in $\mathbb{H}$ of the forms $|w-r|=r$ $(0<r)$ and $\operatorname{Re} w = p$ $(p>0)$, respectively. In particular, the image $f_a(\mathbb{H})$ is a hyperbolic triangle in $\mathbb{H}$. Since we know that these two geodesics form an angle of $\theta=|1-2a|\pi$, we find that $r(1+\cos\theta)=p$ by elementary geometry. Thus, it is enough to show that $p=\sin(\pi a)$, which leads to the relation $r=\sin(\pi a)/(1+\cos((1-2a)\pi)) = 1/(2\sin(\pi a))$.

In order to make statements precise, we introduce some notation. Let f be an analytic function defined in $\mathbb{C}\setminus\mathbb{R}$. For each $x\in\mathbb{R}$, we denote by $f^{\pm}(x)$ the limit $\lim_{t\to 0+} f(x\pm it)$ (if it exists). If f extends analytically to a neighborhood V of x as a single-valued function on $(\mathbb{C}\setminus\mathbb{R})\cup V$, then we write simply $f(x)$ as usual instead of $f^+(x)=f^-(x)$.

With this notation, using [7, (3), (27), pp. 105, 106], we obtain the transformation formulas

$$\frac{2}{\pi}\mathrm{K}_a(-x) = F(a, 1-a; 1; -x) = (1+x)^{-a}F(a, a; 1; \tfrac{x}{1+x}) \tag{4.2}$$

and

$$\begin{aligned}\frac{2}{\pi}\mathrm{K}_a^{\pm}(1+x) &= F^{\pm}(a, 1-a; 1; 1+x) \\ &= (1+x)^{-a}\left[\frac{\Gamma(a)}{\Gamma(2a)\Gamma(1-a)}F(a, a; 2a; \tfrac{1}{1+x}) - e^{\mp\pi a i}F(a, a; 1; \tfrac{x}{1+x})\right],\end{aligned} \tag{4.3}$$

for all $x > 0$. Therefore,

$$f_a^+(-x) = i\frac{\mathrm{K}_a^-(1+x)}{\mathrm{K}_a^+(-x)} = \frac{i\,\Gamma(a)}{\Gamma(2a)\Gamma(1-a)} \cdot \frac{F(a, a; 2a; \frac{1}{1+x})}{F(a, a; 1; \frac{x}{1+x})} - i\,e^{\pi a\, i}, \quad x > 0. \tag{4.4}$$

Thus $\operatorname{Re} f_a^+(-x) = \sin(\pi a)$ for $x > 0$, as required. □

4.5. *Remark.* Since $f_a^+(x) = f_{1-a}^+(x)$, as a by-product of (4.4), we obtain the relation

$$\begin{aligned}&\frac{\Gamma(a)}{\Gamma(2a)\Gamma(1-a)} \cdot \frac{F(a, a; 2a; \frac{1}{1+x})}{F(a, a; 1; \frac{x}{1+x})} - \cos(\pi a) \\ &= \frac{\Gamma(1-a)}{\Gamma(2-2a)\Gamma(a)} \cdot \frac{F(1-a, 1-a; 2-2a; \frac{1}{1+x})}{F(1-a, 1-a; 1; \frac{x}{1+x})} + \cos(\pi a)\end{aligned}$$

for $0 < a < 1$ and $x > 0$. This is equivalent to the identity

$$\begin{aligned}&\frac{\Gamma(a)}{\Gamma(2a)\Gamma(1-a)} \cdot F(a, a; 2a; 1-x)F(1-a, 1-a; 1; x) \\ &- \frac{\Gamma(1-a)}{\Gamma(2-2a)\Gamma(a)} \cdot F(1-a, 1-a; 2-2a; 1-x)F(a, a; 1; x) \\ &- 2\cos\pi a \cdot F(a, a; 1; x)F(1-a, 1-a; 1; x) = 0.\end{aligned}$$

As far as we know, this is a new identity for hypergeometric functions.

Let $\alpha = |1-2a|$. Since f_a maps each of the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, +\infty)$ onto a hyperbolic geodesic segment in $\mathbb{H}$, the pull-back $f_a^*\rho_{\mathbb{H}} = \rho_{\mathbb{H}}(f_a(z))|f_a'(z)||dz|$ of the hyperbolic (or Poincaré) metric $\rho_{\mathbb{H}}(z)|dz| = |dz| /(2\operatorname{Im}(z))$ of the upper half plane $\mathbb{H}$, together with its reflection $f_a^*\rho_{\mathbb{H}}(\bar{z})|dz|$ defines a smooth conformal metric on $\mathbb{C}\setminus\{0, 1\}$. This is the hyperbolic metric $\rho_\alpha(z)|dz|$ of the twice-punctured sphere $\widehat{\mathbb{C}}\setminus\{0, 1\}$ with a conical singularity of angle $2\pi\alpha$ at ∞ (cf. [16, Lemma 2]). We emphasize that the curvature equation, which is equivalent to (2.1),

$$\varDelta \log \rho_\alpha = 4{\rho_\alpha}^2 \tag{4.6}$$

plays an important role in investigation of the metric.

Agard [2] gave a formula for $\rho_{\mathbb{C}\setminus\{0,1\}} = \rho_0$ in terms of complete elliptic integrals. In the same way, we can compute ρ_α for $0 \le \alpha < 1$ with the help of the above construction.

4.7. **Theorem.** *Let $0 \le \alpha < 1$ and choose $0 < a < 1$ so that $\alpha = |1-2a|$. The hyperbolic metric $\rho_\alpha(z)|dz|$ of the twice-punctured sphere $\widehat{\mathbb{C}}\setminus\{0,1\}$ with conical singularity of angle $2\pi\alpha$ at ∞ is given by*

$$\rho_\alpha(z) = \frac{\pi\cos(\pi\alpha/2)}{8|z(1-z)|\operatorname{Re}\left(\mathrm{K}_a(z)\mathrm{K}_a(1-\bar{z})\right)}. \tag{4.8}$$

Proof. By Gauss' contiguous relations (see (2.5.8) of [5]), one obtains

$$z(1-z)\mathrm{K}_a'(z) = (1-a)\big[\mathrm{E}_a(z) - (1-z)\mathrm{K}_a(z)\big].$$

Using this identity, we derive

$$\begin{aligned} f_a'(z) &= -i\,\frac{\mathrm{K}_a'(1-z)\mathrm{K}_a(z) - \mathrm{K}_a(1-z)\mathrm{K}_a'(z)}{(\mathrm{K}_a(z))^2} \\ &= -i\,\frac{1-a}{z(1-z)}\cdot\frac{\mathrm{E}_a^*(z)\mathrm{K}_a(z) + \mathrm{K}_a^*(z)\mathrm{E}_a(z) - \mathrm{K}_a^*(z)\mathrm{K}_a(z)}{(\mathrm{K}_a(z))^2} \\ &= -i\,\frac{\pi\sin(\pi a)}{4z(1-z)(\mathrm{K}_a(z))^2}, \end{aligned}$$

where we have used (3.4). Hence, using the relation $\overline{\mathrm{K}_a(z)} = \mathrm{K}_a(\bar{z})$, we obtain

$$\rho_\alpha(z) = \frac{|f_a'(z)|}{2\operatorname{Im} f_a(z)} = \frac{\pi\sin(\pi a)}{8|z(1-z)(\mathrm{K}_a(z))^2|\operatorname{Re}\left(\mathrm{K}_a(1-\bar{z})/\mathrm{K}_a(\bar{z})\right)},$$

from which the required formula follows. $\square$

By the representation formula for ρ_α, we have the following.

4.9. **Corollary.** *The quantity $\rho_\alpha(z)$ is jointly continuous in α and z.*

Because the formula

$$\mathrm{K}_a(\tfrac{1}{2}) = \frac{\Gamma(\frac{1-a}{2})\Gamma(\frac{a}{2})\sin(\pi a)}{4\sqrt{\pi}} \tag{4.10}$$

is known (see, for instance, [4, (4.5)]), we have the following consequence.

4.11. **Corollary.**

$$\rho_\alpha(\tfrac{1}{2}) = \frac{8\pi^2}{\left(\Gamma(\frac{1+\alpha}{4})\right)^2\left(\Gamma(\frac{1-\alpha}{4})\right)^2\cos(\frac{\pi\alpha}{2})}.$$

The explicit formula in (4.8) of ρ_α can be used to determine the constant terms of asymptotic expansions of ρ_α around singularities.

4.12. **Theorem.** *For $0 < \alpha < 1$, the metric ρ_α satisfies*

$$\log \rho_\alpha(z) = \begin{cases} \log \frac{1}{|z|} - \log\log \frac{1}{|z|} - \log 2 + o(1) & \text{as } z \to 0, \\ \log \frac{1}{|z-1|} - \log\log \frac{1}{|z-1|} - \log 2 + o(1) & \text{as } z \to 1, \\ -(1+\alpha)\log|z| + \log \frac{(\Gamma(\frac{1+\alpha}{2}))^2\Gamma(1-\alpha)}{(\Gamma(\frac{1-\alpha}{2}))^2\Gamma(\alpha)} + o(1) & \text{as } z \to \infty. \end{cases}$$

Proof. Choose $a \in (0, 1/2]$ so that $1 - 2a = \alpha$. First we investigate $\rho_\alpha(z)$ around $z = 0$. Since the $O(1)$ term, say $w(z)$, is known to be continuous at $z = 0$ (see [12, Satz 1] or [9, Theorem 1.1]), it suffices to show that $w(0) = \log 2$. By (3.5), for $x > 0$ we have

$$\mathrm{K}_a(x) = \frac{\pi}{2} + O(x) \quad \text{and} \quad \mathrm{K}_a(1-x) = \frac{\sin(\pi a)}{2}\log\frac{1}{x} + O(1)$$

as $x \to 0+$. Substitution of these formulas into (4.8) yields $w(0) = \log 2$ as required. The corresponding result for $z = 1$ follows from the previous one by the symmetry $\rho_\alpha(1-z) = \rho_\alpha(z)$.

Finally, we consider the case $z \to \infty$. By the general property of conical singularities, one has the expression $\log \rho_\alpha(z) = (2a-2)\log|z| + v(z)$, where $v(z)$ is a continuous function near $z = \infty$ (see [12, Satz 1] or [9, Theorem 1.1]). For $x > 0$, by (4.2), (4.3), and (3.5), we have

$$\frac{2}{\pi}\mathrm{K}_a(-x) = \frac{\Gamma(1-2a)}{(\Gamma(1-a))^2}x^{-a}(1+o(1)) \tag{4.13}$$

and

(4.14)

$$\begin{aligned} &\frac{2}{\pi}\operatorname{Re}\mathrm{K}_a^{\pm}(1+x) \\ &= (1+x)^{-a}\left[\frac{\Gamma(a)}{\Gamma(2a)\Gamma(1-a)}F(a,a;2a;\tfrac{1}{1+x}) - \cos(\pi a)\,F(a,a;1;\tfrac{x}{1+x})\right] \\ &= \left[\frac{\Gamma(a)}{\Gamma(2a)\Gamma(1-a)} - \cos(\pi a)\frac{\Gamma(1-2a)}{(\Gamma(1-a))^2}\right] x^{-a}(1+o(1)) \\ &= \frac{\Gamma(a)}{2\Gamma(2a)\Gamma(1-a)}\,x^{-a}(1+o(1)) \\ &= \frac{(\Gamma(a))^2\sin(\pi a)}{2\pi\Gamma(2a)}\,x^{-a}(1+o(1)) \end{aligned}$$

as $x \to +\infty$. Combining (4.13) and (4.14) with (4.8), we see that

$$\rho_\alpha(-x) = \frac{(\Gamma(1-a))^2\Gamma(2a)}{(\Gamma(a))^2\Gamma(1-2a)}\,x^{2a-2}(1+o(1)), \quad x \to +\infty,$$

which implies that $v(\infty) = \log(\Gamma(1-a))^2\Gamma(2a)/\Gamma(a)^2\Gamma(1-2a))$. This is equal to the required constant term. $\square$

Lehto, Virtanen and Väisälä [10] proved the useful inequality $\rho_0(-|z|) \le \rho_0(z)$ for all $z \in \mathbb{C} \setminus \{0,1\}$. Later on, Weitsman [17] proved a monotonicity property of the hyperbolic metric on a circularly symmetric domain, which means that $\rho_0(r\,e^{i\theta})$ is a non-increasing function of θ in $0 < \theta < \pi$ for a fixed $r > 0$ for the particular domain $\mathbb{C} \setminus \{0,1\}$. We can deduce the same result for ρ_α by employing the method developed in [10].

4.15. **Theorem.** *For $0 \le \alpha < 1$ and fixed $r > 0$, $\rho_\alpha(r\,e^{i\theta})$ is a non-increasing (non-decreasing) function of θ in $0 < \theta < \pi$ $(-\pi < \theta < 0)$. In particular, the inequalities $\rho_\alpha(-|z|) \le \rho_\alpha(z) \le \rho_\alpha(|z|)$ hold for each $z \in \mathbb{C} \setminus \{0,1\}$.*

Proof. It is enough to show the assertion by assuming that $0 < a < \frac{1}{2}$. (The case $a = \frac{1}{2}$ can be treated similarly with the special relation $\rho_0(1/z) = \rho_0(z)|z|^2$ being taken into account.) By the obvious symmetry $\rho_\alpha(\bar{z}) = \rho_\alpha(z)$, it is enough to prove the inequality $\rho_\alpha(r\,e^{i\theta_1}) \ge \rho_\alpha(r\,e^{i\theta_2})$ for $0 \le \theta_1 < \theta_2 \le \pi$. Let $\lambda_1(z) = \rho_\alpha(e^{-i\theta_0}z)$ and $\lambda_2(z) = \rho_\alpha(e^{i\theta_0}z)$, where $\theta_0 = (\theta_1+\theta_2)/2$. Consider now the function $h(z) = \log\lambda_1(z) - \log\lambda_2(z)$. Then h is smooth in $\mathbb{C} \setminus \{0, e^{i\theta_0}, e^{-i\theta_0}\}$ and, by the above symmetry, $h = 0$ on $\mathbb{R} \setminus \{0\}$.

We will show that $h(z) \ge 0$ for $z \in \mathbb{H}$. To this end, we first observe the asymptotic behavior of $h(z)$. It is easy to see that $h(z) \to +\infty$ as $z \to e^{i\theta_0}$. By Theorem 4.12, we also have $h(z) \to 0$ as $z \to \infty$ or $z \to 0$. Therefore, the set $\{z \in \mathbb{H} \setminus \{e^{i\theta_0}\} : h(z) \le -\varepsilon\}$ is compact for each $\varepsilon > 0$. Suppose now that $h < 0$ somewhere in $\mathbb{H}$. Then, there would be a minimum point z_0 for h in $\mathbb{H} \setminus \{e^{i\theta_0}\}$. Then $\Delta h(z_0) \ge 0$ by minimality. On the other hand, the inequality $h(z_0) < 0$ would imply $\lambda_1(z_0) < \lambda_2(z_0)$. Hence, by (4.6),

$$\Delta h(z_0) = 4\lambda_1(z_0)^2 - 4\lambda_2(z_0)^2 < 0,$$

which would be impossible. Thus, we have shown that $h(z) \ge 0$ for $z \in \mathbb{H}$. We now take the point $z_0 = r\,e^{i(\theta_2-\theta_1)/2}$. Then $0 \le h(z_0) = \log\rho_\alpha(r\,e^{-i\theta_1}) - \log\rho_\alpha(r\,e^{i\theta_2}) = \log\rho_\alpha(r\,e^{i\theta_1}) - \log\rho_\alpha(r\,e^{i\theta_2})$, and thus, $\rho_\alpha(r\,e^{i\theta_1}) \ge \rho_\alpha(r\,e^{i\theta_2})$. $\square$

The hyperbolic distance on $\widehat{\mathbb{C}} \setminus \{0,1\}$ induced by ρ_α is defined, as usual, by

$$d_\alpha(z_1, z_2) = \inf_\gamma \int_\gamma \rho_\alpha(z)|dz|,$$

where γ runs over all the rectifiable paths γ connecting z_1 and z_2 in $\widehat{\mathbb{C}} \setminus \{0,1\}$.

As a corollary of Theorem 4.15, we derive a lower estimate for the hyperbolic distance.

4.16. **Corollary.** *For $0 < a < 1$ and $z_1,\ z_2 \in \widehat{\mathbb{C}} \setminus \{0,1\}$ with $|z_1| \le |z_2|$, the following inequality holds:*

$$d_\alpha(z_1, z_2) \ge d_\alpha(-|z_1|, -|z_2|) = \int_{|z_1|}^{|z_2|} \rho_\alpha(-t)dt. \tag{4.17}$$

We can compute the last integral by the following result.

4.18. **Theorem.** *Let* $\alpha = |1-2a|$ *for* $0 < a < 1$. *The formula*

$$\int_x^y \rho_\alpha(-t)dt = \Phi_a(y) - \Phi_a(x)$$

holds for $0 < x < y$, *where*

$$\Phi_a(x) = -\frac{1}{2}\log\left(\frac{\Gamma(a)}{\Gamma(2a)\Gamma(1-a)}\frac{F(a,a;2a;\frac{1}{1+x})}{F(a,a;1;\frac{x}{1+x})} - \cos(\pi a)\right).$$

Proof. One can proceed almost as in the proof of [15, Lemma 5.1]. We can write $f_a^+(-x)$ in the form $i\,u(x) + \sin(\pi a)$ for $x > 0$ by (4.4). Since $\rho_\alpha(-t) = |(f_a^+)'(-t)|/2\,\mathrm{Im}\, f_a^+(-t) = -u'(t)/2u(t)$, we obtain

$$\int_x^y \rho_\alpha(-t)dt = -\int_x^y \frac{u'(t)}{2u(t)}dt = \frac{1}{2}\log\frac{u(x)}{u(y)} = \Phi_a(y) - \Phi_a(x).$$

□

Note that when $a \neq \frac{1}{2}$,

$$\Phi_a(\infty) = -\frac{1}{2}\log\cos(\pi a)$$

is positive and finite, whereas $\Phi_{1/2}(\infty) = \infty$.

4.19. *Remark.* More generally, the ρ_α-distance between z_1 and z_2 in $\overline{\mathbb{H}} \setminus \{0,1,\infty\}$ can be expressed by

$$d_\alpha(z_1,z_2) = \operatorname{arctanh}\left|\frac{f_a(z_2) - f_a(z_1)}{f_a(z_2) - \overline{f_a(z_1)}}\right|,$$

where $0 < a < 1$ is chosen so that $\alpha = |1-2a|$ and f_a is given in Lemma 4.1. Indeed, by construction, f_a is an isometric embedding of $(\mathbb{H}, \rho_\alpha)$ into $(\mathbb{H}, \rho_{\mathbb{H}})$ and its image Δ_a is (hyperbolically) convex in $\mathbb{H}$. Therefore, the geodesic segment joining z_1 and z_2 in $\widehat{\mathbb{C}} \setminus \{0,1\}$ with respect to ρ_α is contained in the closure of $\mathbb{H}$ and its image under f_a is the hyperbolic geodesic joining $f_a(z_1)$ and $f_a(z_2)$. It is well known that the hyperbolic distance between two points w_1 and w_2 in $\mathbb{H}$ is given by $\operatorname{arctanh}|(w_2-w_1)/(w_2-\overline{w_1})|$, and the above formula follows.

Finally, we mention monotonicity of $\rho_\alpha(z)$ with respect to the parameter α.

4.20. **Proposition.** *The density* $\rho_\alpha(z)$ *is non-increasing in* $0 \le \alpha < 1$ *for a fixed* $z \in \mathbb{C} \setminus \{0,1\}$.

Though this result is contained in [14, Prop. 2.4] as a special case, we give a proof for convenience of the reader. The assertion is established by a simple application of the Schwarz-Pick-Ahlfors lemma (cf. [3]). Here, we employ the same technique as in Theorem 4.15.

Proof. For a given pair α, α' with $0 \le \alpha < \alpha' < 1$, we consider the function $h = \log \rho_{\alpha'} - \log \rho_\alpha$ in $\mathbb{C} \setminus \{0,1\}$. By Theorem 4.12, the function h extends continuously to 0 and 1 if we set $h(0) = h(1) = 0$, and has the asymptotic behavior $h(z) = (\alpha - \alpha' + o(1)) \log |z|$ as $z \to \infty$. Therefore, if h takes a positive value, there is a point $z_0 \in \mathbb{C} \setminus \{0,1\}$ at which h attains its (positive) maximum. Then $\Delta h(z_0) \le 0$. On the other hand, by (4.6),

$$\Delta h(z_0) = 4\rho_{\alpha'}(z_0)^2 - 4\rho_\alpha(z_0)^2 > 0,$$

which is a contradiction. Hence, we conclude that $h(z) \le 0$, in other words, $\rho_\alpha(z) \ge \rho_{\alpha'}(z)$ for $z \in \mathbb{C} \setminus \{0,1\}$. □

4.21. *Remark.* The expression $\rho(a,z) \equiv \rho_{|1-2a|}(z)$ as in (4.8) can be viewed as a smooth function in $(a,z) \in (0,1) \times (\mathbb{C} \setminus \{0,1\})$. Then it has the obvious symmetry $\rho(1-a,z) = \rho(a,z)$. By the above theorem, $\rho(a,z)$ attains its maximum at $a = \frac{1}{2}$ for a fixed z. In particular, by this observation we obtain $(\partial\rho/\partial a)(\frac{1}{2}, z) = 0$. We also see that $\rho_\alpha(z) \to 0$ as $\alpha \to 1$ from (4.8). This corresponds to the well-known fact that the twice-punctured sphere $\widehat{\mathbb{C}} \setminus \{0,1\}$ does not carry a hyperbolic metric.

5. Applications

We conclude the present note with a few applications of our metric ρ_α. Since no concrete estimates for ρ_α are given so far, we will give only general principles to refine classical results.

If a meromorphic function f on the unit disk $\mathbb{D}$ does not assume the three points $0, 1$ and ∞, then the principle of hyperbolic metric gives us the inequality $f^*\rho \le \rho_{\mathbb{D}}$, namely;

$$\rho(f(z))|f'(z)| \le \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.$$

The classical theorems of Picard and Schottky follow essentially from the above inequality (see, for example, [3, §1-9]). We can now relax the assumption about the omitted values as in the following.

5.1. **Theorem.** *Let f be a meromorphic function on the unit disk omitting the two values 0 and 1. Suppose that every pole of f is of order at least $k \ge 2$. Then the following inequality holds:*

$$\rho_{1/k}(f(z))|f'(z)| \le \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Proof. Let $\alpha > 1/k$. We may assume that f is not constant. Let λ be the pull-back metric $f^*\rho_\alpha$ of ρ_α under f. Then it is easily verified that the Gaussian curvature of λ is -4 off the set of poles and branch points of f. Let z_0 be a pole of f. Then the order m of the pole at z_0 is at least k by assumption. In view of (3.5), we have

$$\begin{aligned}\log\lambda(z) &= -(1+\alpha)\log|f(z)| + \log|f'(z)| + O(1)\\ &= \big[(1+\alpha)m - (m+1)\big]\log|z-z_0| + O(1)\\ &= (\alpha m - 1)\log|z-z_0| + O(1)\end{aligned}$$

as $z \to z_0$. Since $\alpha m > 1$, we see that $\lambda(z_0) = 0$. Thus λ is an ultrahyperbolic metric on $\mathbb{D}$ in the sense of Ahlfors [3]. Thus, Ahlfors' lemma now yields $f^*\rho_\alpha \le \rho_{\mathbb{D}}$. Taking the limit as $\alpha \to 1/k$, we obtain the required inequality. □

Knowledge about the hyperbolic metric $\rho = \rho_0$ of the thrice-punctured sphere $\mathbb{C}\setminus\{0,1\}$ has led to various useful estimates for the hyperbolic metric of a general plane domain (see, for instance, [6] or [15]). We now use ρ_α instead of ρ_0 to obtain similar estimates for the hyperbolic metric with conical singularities.

5.2. **Theorem.** *Let Ω be a subdomain of the Riemann sphere $\widehat{\mathbb{C}}$ with $\infty \in \Omega$ such that $\widehat{\mathbb{C}}\setminus\Omega$ contains at least two points. Let λ be a conformal metric on Ω with conical singularities of angle less than 2π. Suppose that λ has a conical singularity of angle $2\pi\alpha > 0$ and that for each $w_0 \in \partial\Omega$, $|z-w_0|\log(1/|z-w_0|)\lambda(z)$ is bounded away from 0 in $V\cap\Omega$ for a neighborhood V of w_0 if w_0 is isolated in $\partial\Omega$ and $|z-w_0|\log(1/|z-w_0|)\lambda(z) \to +\infty$ as $z \to w_0$ in Ω otherwise. Then*

$$\lambda(z) \ge \sup_{w_0,w_1\in\partial\Omega} \frac{1}{|w_1-w_0|}\rho_\alpha\left(\frac{z-w_0}{w_1-w_0}\right), \quad z \in \Omega\setminus\{\infty\}.$$

Proof. We follow the argument used by Heins [8, §20]. First note that the set S of conical singularities of λ can be characterized as $\{z \in \Omega\setminus\{\infty\} : \lambda(z) = \infty\} \cup \{\infty\}$. Pick $\alpha' \in (\alpha, 1)$ and fix a pair of distinct points w_0 and w_1 in $\partial\Omega$. Set $\mu(z) = \rho_{\alpha'}((z-w_0)/(w_1-w_0))/|w_1-w_0|$ and let

$$v = \max\{\log\mu - \log\lambda,\ 0\}.$$

Then v is subharmonic on $\Omega\setminus S$ and vanishes in a neighborhood of S. Moreover, $v = 0$ near every boundary point of Ω except possibly for w_0 and w_1. If w_j is not isolated, then this is still valid. Otherwise, by the local behavior of solutions to the Liouville equation around an isolated singularity due to Nitsche [12], we see that v can be extended continuously to the point w_j. Recall now the following fact: *Suppose that u is a continuous function on an open neighborhood D of a point a and subharmonic on $D\setminus\{a\}$. Then u is subharmonic on D.* Thus v is subharmonic on $\Omega' = \Omega\cup\{w_j : w_j \text{ is isolated}\}$ and vanishes in a neighborhood of $\partial\Omega'\cup S$. We now appeal to the maximum

principle to conclude that $v = 0$ in Ω, which means $\mu \le \lambda$. The proof is now complete. $\square$

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1972.

[2] S. Agard, *Distortion theorems for quasiconformal mappings*, Ann. Acad. Sci. Fenn. A I Math. **413** (1968), 1–12.

[3] L. V. Ahlfors, *Conformal Invariants*, McGraw-Hill, New York, 1973.

[4] G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen, *Generalized elliptic integrals and modular equations*, Pacific J. Math. **192** (2000), 1–37.

[5] G. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge U. Press, 1999.

[6] A. F. Beardon and Ch. Pommerenke, *The Poincaré metric of plane domains*, J. London Math. Soc. (2) **18** (1978), 475–483.

[7] Bateman Manuscript Project (ed. A. Erdélyi and W. Magnus and F. Oberhettinger and F. G. Tricomi), *Higher Transcendental Functions* Vol. I, McGraw-Hill, New York, 1953.

[8] M. Heins, *On a class of conformal metrics*, Nagoya J. Math. **21** (1962), 1–60.

[9] D. Kraus and O. Roth, *The behaviour of solutions of the Gaussian curvature equation near an isolated boundary point*, Math. Proc. Cambridge Philos. Soc. **145** (2008), 643–667.

[10] O. Lehto, K. I. Virtanen, and J. Väisälä, *Contributions to the distortion theory of quasiconformal mappings*, Ann. Acad. Sci. Fenn. A I No. **273** (1959), pp. 14.

[11] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New York,1952.

[12] J. Nitsche, *Über die isolierten Singularitäten der Lösungen von $\Delta u = e^u$*, Math. Z. **68** (1957), 316–324.

[13] E. Picard, *De l'intégration de l'équation $\Delta u = e^u$ sur une surface de Riemann fermée*, J. Reine Angew. Math. **130** (1905), 243–258.

[14] G. Schumacher and S. Trapani, *Variation of cone metrics on Riemann surfaces*, J. Math. Anal. Appl. **311** (2005), 218–230.

[15] T. Sugawa and M. Vuorinen, *Some inequalities for the Poincaré metric of plane domains*, Math. Z. **250** (2005), 885–906.

[16] L. Takhtajan and P. Zograf, *Hyperbolic 2-spheres with conical singularities, accessory parameters and Kähler metrics on $\mathcal{M}_{0,n}$*, Trans. Amer. Math. Soc. **355** (2003), 1857–1867.

[17] A. Weitsman, *A symmetric property of the Poincaré metric*, Bull. London Math. Soc. **11** (1979), 295–299.

ANDERSON:
Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA
email: anderson@math.msu.edu
FAX: +1-517-432-1562

SUGAWA:
Graduate School of Information Sciences
Tohoku University

Aoba-ku, Sendai, 980-8579 JAPAN
email: sugawa@math.is.tohoku.ac.jp
FAX: +81-22-795-4654

VAMANAMURTHY:
Department of Mathematics
University of Auckland
Auckland, NEW ZEALAND
email: vamanamu@math.auckland.nz
FAX: +649-373-7457

VUORINEN:
Department of Mathematics
University of Turku
Vesilinnantie 5
FIN-20014, FINLAND
e-mail: vuorinen@utu.fi
FAX: +358-2-3336595